Two-dimensional pentagonal structures in dissipative systems

T. Frisch and G. Sonnino
Institut de Non Linéaire de Nice, 1361 Route des Lucioles, 06560 Valbonne, France
(Received 13 September 1993; revised manuscript received 8 August 1994)

We report the existence and stability of a two-dimensional pattern with a tenfold orientational order in a dissipative system described by partial differential equations. The pattern appears in a subcritical way and results from a superposition of two linearly unstable patterns with different wave numbers.

PACS number(s): 47.20.Ky, 61.44.+p, 61.50.Em, 64.70.Pf

I. INTRODUCTION

Pattern formation in dissipative systems has been widely studied in the recent years both from the experimental and theoretical point of view. Such patterns are observed in a whole variety of natural phenomena such as in convecting fluid, liquid crystals, optical systems, and chemistry. For these cases the problem of interest is whether there exists a universal basic mechanism of transition from the spatially homogeneous pattern to a perfectly ordered cellular pattern. Experimentally, a wide variety of structurally ordered patterns has been observed (rolls, rhombi, hexagons, etc.). Some years ago, quasiperiodic crystals have been experimentally observed in condensed matter physics [1–4] and more recently, also in dissipative system [5]. In this experiment, a stable standing wave pattern with a twelvefold orientational order is generated by the parametric excitation of capillary waves. The excitation spectrum consisted of two frequencies $(m\nu, n\nu)$ with m and n relatively prime and of different parity. In this last example, the pattern wave number is fixed by the excitation frequency and can be understood as superposition of two linearly unstable wave patterns. From a theoretical point of view, it is natural to explain the existence of quasiperiodic structures in the framework of amplitude equations [6–8]. In Ref. [7], pentagonal structure were found stable using amplitude equations not model equations.

In this paper, we present a simple model of a dissipative system whose dynamics is described by two coupled Swift-Hohenberg equations. We shall consider a spatially extended system so that the influence of the boundaries can be neglected. Here, in analogy with the experimental work reported in Ref. [5], two different wave numbers can become unstable at the same time and the structure results from a superposition of two linearly unstable patterns with different wave numbers. We shall show that there exists a region in the parameters space where patterns with a tenfold orientational order are stable.

II. PATTERNS WITH FIVEFOLD ORIENTATIONAL ORDER IN COUPLED SWIFT-HOHENBERG EQUATIONS

We start by studying a model whose dynamics is described by two coupled Swift-Hohenberg Eqs. [9].

$$\partial_t \Psi_1 = [\varepsilon\mu_1 - (\nabla^2 + k_0^2)^2]\Psi_1 + \varepsilon^{1/2} a \Psi_1 \Psi_2 + \varepsilon^{1/2} b \Psi_2^2 - \alpha \Psi_1^3 , \quad (1)$$

$$\partial_t \Psi_2 = [\varepsilon\mu_2 - (\nabla^2 + q_0^2)^2]\Psi_2 + \varepsilon^{1/2} c \Psi_1 \Psi_2 + \varepsilon^{1/2} d \Psi_1^2 - \alpha \Psi_2^3 . \quad (2)$$

Our aim is to show the existence and stability of pentagonal structures. To this end, we assume control parameter ε to be small. As can be easily checked, when μ_1 and μ_2 become positive, the homogeneous solution loses its stability toward a structured solution. Near the transition, due to the rotational invariance of the system, the structure of all modes which are equally amplified is $e^{i\mathbf{k}\cdot\mathbf{r}}$. Each $\mathbf{k}$ lies in two separate rims of radius k_0 and q_0, respectively. Pentagonal patterns corresponding to the final state in which only a finite number of modes (five) participate are

$$\Psi_1 = \sum_{j=1}^{5} A_j e^{i\mathbf{q}_j\cdot\mathbf{r}} + \text{c.c.} + \text{h.o.t.} , \quad (3)$$

$$\Psi_2 = \sum_{j=1}^{5} B_j e^{i\mathbf{k}_j\cdot\mathbf{r}} + \text{c.c.} + \text{h.o.t.} , \quad (4)$$

where vectors k_j and q_j are the unstable wave numbers shown in Fig. 1. "c.c." and "h.o.t." stand for complex conjugate and higher order terms, respectively. Note that, to avoid hexagonal patterns, Eq. (1) does not contain quadratic terms of the form Ψ_1^2 as does Eq. (2). Again, we can easily convince ourselves that the two pentagonal patterns will resonate only if the following relations among vectors $\mathbf{k}_0^i$ and $\mathbf{q}_0^i$ are satisfied (see Fig. 2)

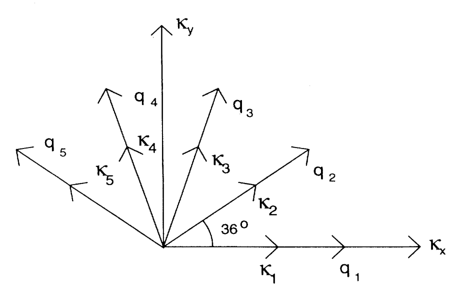

FIG. 1. The pentagonal pattern results from a superposition of two linearly unstable structures with different wave numbers.

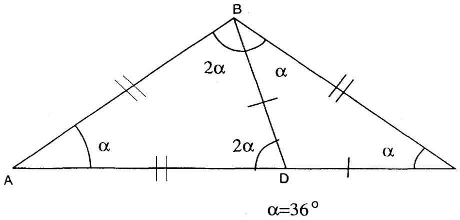

FIG. 2. The picture illustrates which relations have to be satisfied among the lengths of the side of the triangle in order to obtain a pentagonal structure: $|AB|=|BC|=|DA|$ and $|DB|=|CD|$.

$$\mathbf{q}_i=\mathbf{q}_{i+1}-\mathbf{k}_{i+3}\ , \tag{5}$$

$$\mathbf{q}_i=\mathbf{k}_{i+1}-\mathbf{k}_{i+4}\ . \tag{6}$$

The index i runs from 1 to 10. Of course, previous relations holds only if the two unstable wave numbers k_0 and q_0 satisfy the relation $k_0/q_0=2\sin(\pi/10)$. It is easy to verify that, near transition, the equations of the critical amplitudes A_i and B_i read

$$\begin{aligned}\partial_t A_i=&\mu_1 A_i+a(A_{i+1}\overline{B}_{i+3}+\overline{A}_{i+4}B_{i+2})+2bB_{i+1}\overline{B}_{i+4}\\&-3\alpha A_i[|A_i|^2+2(|A_{i+1}|^2+|A_{i+2}|^2\\&+|A_{i+3}|^2+|A_{i+4}|^2)]\ ,\end{aligned} \tag{7}$$

$$\begin{aligned}\partial_t B_i=&\mu_2 B_i+c(B_{i+3}\overline{A}_{i+4}+\overline{B}_{i+2}A_{i+1})+2dA_{i+2}\overline{A}_{i+3}\\&-3\alpha B_i[|B_i|^2+2(|B_{i+1}|^2+|B_{i+2}|^2\\&+B_{i+3}|^2+|B_{i+4}|^2)]\ ,\end{aligned} \tag{8}$$

where the index i runs from 1 to 5.

Note that owing to the phase invariance, critical amplitudes A_i and B_i can be chosen are real. For simplicity, let us now study the case where the dynamics of the model is described by a free energy. This occurs when particular relations among parameters a,b,c,d are satisfied: $a=2b=c=2d$. The expression of the free energy reads

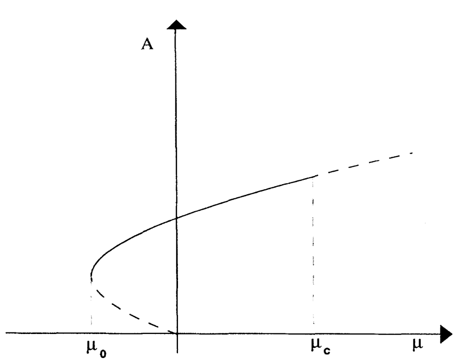

FIG. 3. The bifurcation diagram for pentagonal structures: the solid line denotes the region where the "quasistructure" is stable.

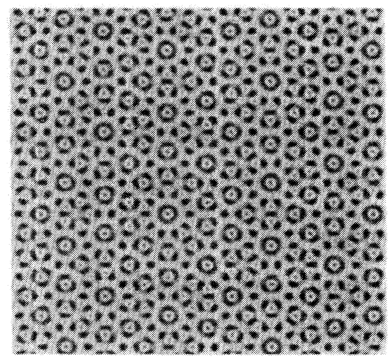

FIG. 4. Pentagonal pattern with a tenfold orientational order. This picture has been obtained by simulating Eqs. (1) and (2) and showing the levels of Ψ_2. The numerical values are $\varepsilon=0.1$, $\mu_1=\mu_2=0$, $\alpha=1$, and $a=2b=c=2d=1$.

$$\begin{aligned}F=&-\tfrac{1}{2}\varepsilon\mu_1\Psi_1^2-\tfrac{1}{2}\varepsilon\mu_2\Psi_2^2+\tfrac{1}{2}[(\nabla^2+k_0^2)(\Psi_1)^2]\\&+\tfrac{1}{2}[(\nabla^2+q_0^2)(\Psi_2)^2]\\&-\varepsilon^{1/2}a\Psi_1\Psi_2(\Psi_1+\Psi_2)+\frac{\alpha}{4}(\Psi_1^4+\Psi_2^4)\ .\end{aligned} \tag{9}$$

The fixed point of Eqs. (7) and (8) (corresponding to pentagons) can be found by setting $A_i=A_j$ and $B_i=B_j$. When $\mu_1=\mu_2$, we obtain

$$A_i=B_i=\frac{a\pm\sqrt{a^2+12\alpha\mu}}{18\alpha}\ . \tag{10}$$

As known, the stability of the pentagonal patterns is guaranted if the eigenvalues of the linearized operator around the fixed point are all negatives. An important feature of this problem is the cyclic (by blocks) nature of the matrix. Thanks to this property, we easily find the range of stability of the pentagonal structure (see Fig. 3) as being

$$\mu_0<\mu<\mu_c \tag{11}$$

where μ_0 and μ_c are, respectively,

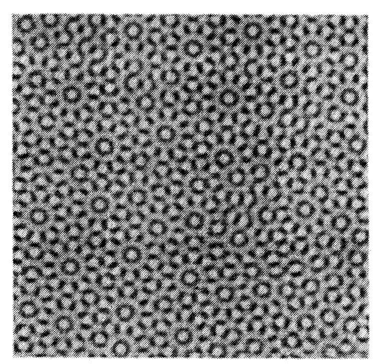

FIG. 5. Pentagonal pattern with a tenfold orientational order showing the presence of defects. This picture has been obtained by simulating Eqs. (1) and (2) and showing the levels of Ψ_2. The numerical values are $\varepsilon=0.1$, $\mu_1=\mu_2=0$, $\alpha=1$, and $a=2b=c=2d=1$.

$$\mu_0 = \frac{-a^2}{12\alpha} \quad \text{and} \quad \mu_c = \frac{55a^2}{108\alpha} . \tag{12}$$

In Figs. 4 and 5, two numerical simulations (256×256 points) of Eqs. (1) and (2) are reported. Figure 5 still shows a pentagonal structure even though many defects exist. Note that numerical simulations have confirmed the stability of pentagons even when the dynamics of model is not governed by a free energy.

III. CONCLUSIONS

We have presented a model that foresees a first-order transition from a spatially homogeneous pattern to a pattern with tenfold orientational order. The "quasicrystalline" structure is stable in a large region of the parameter space. This result has been obtained by setting the parameters in such a way that Eqs. (1) and (2) can be written in a variational form. As shown, the stability of the pentagonal structure can be easily understood by simple geometrical arguments.

ACKNOWLEDGMENTS

It is a pleasure to thank P. Coullet and S. Rica for stimulating discussions and C. Green for having carefully read this manuscript. The work of G. Sonnino has been sponsored by an EEC Grant (Contract No. ERBSC1* CT005143). Numerical simulations have been done by the Connection Machine of the Inria (Institut National de Recherche and Informatique et Automatique) at Sophia-Antipolis.

[1] D. Schectman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).
[2] T. Ishimasa, H. U. Nissen, and Y. Fukano, Phys. Rev. Lett. **55**, 511 (1985).
[3] L. Bendersky, Phys. Rev. Lett. **55**, 1461 (1985).
[4] N. Wang, H. Chen, and K. H. Kuo, Phys. Rev. Lett. **59**, 1010 (1987).
[5] W. S. Edwards and S. Fauve, C. R. Acad. Sci. Ser. II, **315**, 417 (1992); W. S. Edwards and S. Fauve, Phys. Rev. E **47**, 788 (1993).
[6] N. D. Mermin and S. M. Troian, Phys. Rev. Lett. **54**, 1525 (1985).
[7] B. A. Malomed, A. A. Nepomnyashchii, and M. I. Tribelskii, Zh. Eksp. Teor. Fiz. **96**, 684 (1989) [Sov. Phys. JETP **69**, 388 (1989)].
[8] A. C. Newell and Y. Pomeau (unpublished).
[9] J. B. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

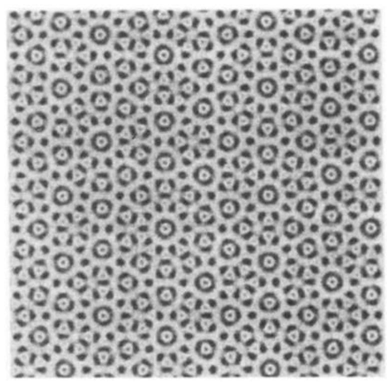

FIG. 4. Pentagonal pattern with a tenfold orientational order. This picture has been obtained by simulating Eqs. (1) and (2) and showing the levels of Ψ_2. The numerical values are $\varepsilon=0.1$, $\mu_1=\mu_2=0$, $\alpha=1$, and $a=2b=c=2d=1$.

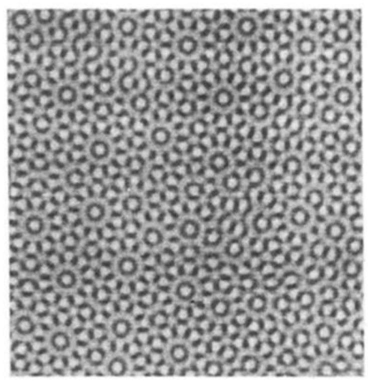

FIG. 5. Pentagonal pattern with a tenfold orientational order showing the presence of defects. This picture has been obtained by simulating Eqs. (1) and (2) and showing the levels of Ψ_2. The numerical values are $\varepsilon=0.1$, $\mu_1=\mu_2=0$, $\alpha=1$, and $a=2b=c=2d=1$.